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Application of polynomial method to on-line list colouring of graphs

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ABSTRACT

A graph is on-line chromatic choosable if its on-line choice number equals its chromatic number. In this paper, we consider on-line chromatic-choosable complete multi-partite graphs. Assume G is a complete k-partite graph. It is known that if the polynomial $P(G)$ defined as $P(G)=\prod_{u<v,uv\in E}(x_u-x_v)$ has one monomial $\prod_{v\in V}x_v^{\varphi(v)}$ with $\varphi(v)<k$ whose coefficient is nonzero, then G is on-line k-choosable. It is usually difficult to calculate the coefficient of a monomial in $P(G)$. For several classes of complete multi-partite graphs G, we introduce different polynomials $Q(G)$ associated to G, such that $Q(G)$ and $P(G)$ have the same coefficient for those monomials of highest degree. However, the calculation of the coefficient of some monomials based on $Q(G)$ is easier. Using this method, we prove the following graphs are on-line k-choosable: $K_{\ell+1,1*(\ell-1),2*(k-\ell)}$, $K_{s,t,1*(k-2)}$ (where $(s-1)(t-1)\leq 2k-3$), $K_{3*2,1*2,2*(k-4)}$ and $K_{4,3,1*3,2*(k-5)}$. These results provide support for the on-line version of Ohba's conjecture: graphs G with $|V(G)|\leq 2\chi(G)$ are on-line chromatic-choosable.

1. Introduction

A *list assignment* of a graph G is a mapping L which assigns to each vertex v a set $L(v)$ of colours. An *L-colouring* of G is a proper vertex colouring c of G such that $c(v)\in L(v)$ for each v. We say G is *L-colourable* if there exists an L-colouring of G. A graph G is called k-choosable if for any list assignment

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L with $|L(v)| = k$ for all $v \in V(G)$, G is L-colourable. More generally, for a mapping $f : V(G) \to \mathbb{N}$ (we use the convention that $\mathbb{N} = \{0, 1, 2, \ldots\}$), we say G is *f-choosable* if for every list assignment L with $|L(v)| = f(v)$, G is L-colourable. The *choice number* $ch(G)$ of G is the minimum k for which G is k-choosable. List colouring of graphs was introduced in the 1970s by Vizing [20] and independently by Erdős et al. [3], and has been studied extensively in the literature [19].

A list assignment of a graph G can be given alternately as follows: Without loss of generality, we may assume that $\cup_{v \in V(G)} L(v) = \{1, 2, \ldots, q\}$ for some integer q. For $i = 1, 2, \ldots, q$, let $V_i = \{v : i \in L(v)\}$. The sequence $(V_1, V_2, \ldots, V_q)$ is just another way of specifying the list assignment. In the following, we shall denote a list assignment L as $L = (V_1, V_2, \ldots, V_q)$. An L-colouring of G is equivalent to a sequence $(X_1, X_2, \ldots, X_q)$ of independent sets that form a partition of $V(G)$ and such that $X_i \subseteq V_i$ for $i = 1, 2, \ldots, q$. If $v \in X_i$, then we say v is coloured with colour i.

This alternate definition motivates the definition of the following list colouring game on a graph G, which was introduced in [15,13].

Definition 1. Given a graph G and a mapping $f : V(G) \to \mathbb{N}$. Two players play the following game. In the ith step, Player A chooses a non-empty subset V_i of $V(G)$, and Player B chooses an independent set X_i contained in V_i. A vertex v is *coloured* if $v \in X_i$ for some i, and is *finished* if v is contained in $f(v)$ of the V_i's. When Player A chooses the set V_i, it is required V_i contains only uncoloured non-finished vertices. If for some integer m, at the end of the mth step, there is a finished vertex v that is uncoloured, then Player A wins the game. Otherwise, at some step, all vertices are coloured. In this case, Player B wins the game.

Thus in the game, Player A is required to give $f(v)$ permissible colours to vertex v and Player B needs to colour v with a permissible colour, under the restriction that no colour is assigned to two adjacent vertices. Player B wins the game if every vertex v is successfully coloured.

The game is called the Painter and Correct game in [15,13]. In some sense, one can view it as an on-line version of a list colouring game: it is the same as a list colouring of a graph, except that the list assignment is given on-line, and the colouring is constructed on-line.

Definition 2. Suppose $f : V(G) \to \mathbb{N}$. We say G is *on-line f-choosable* if Player B has a winning strategy in the f-list colouring game on G.

For positive integers n, G is *on-line n-choosable* means that G is on-line f-choosable for the constant function $f \equiv n$. The *on-line choice number* $\mathrm{ch}^{\mathrm{OL}}(G)$ of G is the minimum n for which G is on-line n-choosable.

It follows from the definition that for any graph G, $\mathrm{ch}^{\mathrm{OL}}(G) \geq ch(G)$. However, it is shown in [7,14,15,13] that many upper bounds for the choice number of a graph remain upper bounds for its on-line choice number. For example, the on-line choice number of planar graphs is at most 5, the on-line choice number of the line graph $L(G)$ of a bipartite graph G is $\Delta(G)$, and if G has an orientation in which the number of even Eulerian subgraphs differs from the number of odd Eulerian subgraphs and $f(x) = d^+(x) + 1$, then G is on-line f-choosable. It is shown in [21] that the on-line choice number of any graph G on p vertices is at most $\chi(G) \ln p + 1$, and there are graphs whose on-line choice number is strictly bigger than its choice number. A characterization of on-line 2-choosable graphs is also given in [21].

For almost any list colouring problem, there is a corresponding on-line list colouring problem. As mentioned above, an upper bound for the choice number of a graph might also be an upper bound for its on-line choice number. However, even if this is true, a proof very much different from the original one may be needed so that it works for the on-line list colouring version.

It follows from the definition that for any graph G, $\chi(G) \leq ch(G) \leq \mathrm{ch}^{\mathrm{OL}}(G)$. A graph G is called *on-line chromatic-choosable* (respectively, *chromatic-choosable*) if $\chi(G) = \mathrm{ch}^{\mathrm{OL}}(G)$ (respectively, $\chi(G) = ch(G)$).

Chromatic-choosable graphs have attracted considerable attention. Conjecture 1 was made independently by Vizing, by Gupta, by Albertson and Collins, and by Bollobas and Harris (see [6] and [8]).

Conjecture 1. *Line graphs are chromatic-choosable.*

More generally, the following conjecture was proposed in [5].

Conjecture 2. *Claw-free graphs (i.e., graphs with no induced $K_{1,3}$) are chromatic-choosable.*

The following special case of Conjecture 1 has been proved by Galvin [4].

Theorem 1 (*Galvin [4]*)**.** *If G is the line graph of a bipartite graph, then G is chromatic-choosable.*

As observed by Schauz [13], the proof of Theorem 1 works for on-line list colouring as well. So we have the following result.

Theorem 2. *If G is the line graph of a bipartite graph, then G is on-line chromatic-choosable.*

A conjecture of Ohba [10] concerning chromatic-choosable graphs also received a lot of attention.

Conjecture 3 (*Ohba [10]*)**.** *If $|V(G)| \le 2\chi(G)+1$, then G is chromatic-choosable.*

Some partial results on the conjecture are obtained. It was proved by Reed and Sudakov [12] that if $|V(G)| \le \frac{5}{3}\chi(G) - \frac{4}{3}$, then G is chromatic-choosable. Ohba [11] showed that if $|V(G)| \le 2\chi(G)$, and G has a $\chi(G)$ colouring with each colour class of size at most 3, then G is chromatic-choosable. This was improved in [17] where the condition $|V(G)| \le 2\chi(G)$ is relaxed to $|V(G)| \le 2\chi(G)+1$. Suppose $k = k_1 + k_2 + \cdots + k_s$, and $n_1, n_2, \ldots, n_s$ are positive integers. We denote by $K_{n_1*k_1, n_2*k_2, \ldots, n_s*k_s}$ the complete k-partite graph in which k_i parts are of cardinality n_i for $i = 1, 2, \ldots, s$. If $k_i = 1$, then $n_i * 1$ in the subscript will be shortened as n_i (for example $K_{3,2*3} = K_{3*1,2*3}$). Ohba's conjecture has also been confirmed for the following graphs (see [18]):

$$K_{3*2,2*(k-3),1}, \quad K_{3,2*(k-1)}, \quad K_{s+3,2*(k-s-1),1*s}, \quad K_{4,3,2*(k-4),1*2}, \quad K_{5,3,2*(k-5),1*3}.$$
$$K_{3*3,2*(k-5),1*2}, \quad K_{4,3*2,2*(k-6),1*3}.$$

In this paper, we are interested in the the on-line version of Ohba's conjecture. It is proved in [9] that for $n \ge 2$, the graph $K_{3,2*n}$ is not on-line $(n+1)$-choosable. So the best one can hope is the following:

Conjecture 4. *If $|V(G)| \le 2\chi(G)$, then G is on-line chromatic-choosable.*

It turns out that all the proofs of the special cases of Ohba's conjecture mentioned above use Hall's Theorem to obtain a matching between vertices and colours under certain conditions. This means that one needs to know the whole list assignment before colouring the vertices. Therefore the proofs do not work for on-line list colouring. We shall use Combinatorial Nullstellensatz to prove that $K_{\ell+1,1*(\ell-1),2*(k-\ell)}$, $K_{s,t,1*(k-2)}$ (where $(s-1)(t-1) \le 2k-3$), $K_{3*2,1*2,2*(k-4)}$ and $K_{4,3,1*3,2*(k-5)}$ are on-line k-choosable.

2. Polynomial method

Suppose G is a graph. We associate to each vertex v a variable x_v. Assume the vertices of G are linearly ordered. Let $P(G)$ be the polynomial defined as $P(G) = \prod_{u<v, uv\in E}(x_u - x_v)$. A colouring $c : V(G) \to \{1, 2, \ldots\}$ is a proper vertex colouring of G if and only if $P(G, c) \ne 0$, where $P(G, c)$ is the evaluation of $P(G)$ at $x_v = c(v)$ for all $v \in V(G)$. Combinatorial Nullstellensatz [2] gives a sufficient condition for the existence of an assignment c with fixed image range so that $P(G, c) \ne 0$. For a mapping $\varphi : V(G) \to \{0, 1, \ldots\}$, let $[P(G)]_\varphi$ be the coefficient of the monomial $\prod_{v\in V(G)} x_v^{\varphi(v)}$ in the expansion of $P(G)$. Combinatorial Nullstellensatz proved by Alon and Tarsi [1] implies that if $\sum_{v\in V(G)} \varphi(v)$ is equal to the degree of $P(G)$ and $[P(G)]_\varphi \ne 0$, then G is $(\varphi+1)$-choosable, i.e., for any list assignment L of G which assigns to v a set $L(v)$ of $\varphi(v)+1$ permissible colours, G has a proper L-colouring. This result is strengthened by Schauz in [15] that the graph G is on-line $(\varphi+1)$-choosable.

In this paper, we use this method to prove that if $G = K_{\ell+1,1*(\ell-1),2*(k-\ell)}$, $K_{s,t,1*(k-2)}$ (where $(s-1)(t-1) \le 2k-3$), $K_{3*2,1*2,2*(k-4)}$ or $K_{4,3,1*3,2*(k-5)}$, then G is on-line k-choosable. For this purpose, we need to show that $[P(G)]_\varphi \ne 0$ for some mapping φ.

To calculate the coefficient of a monomial in a polynomial, we shall use the following lemma proved in [16].

Lemma 3. *If $P(x_1, x_2, \ldots, x_n) \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ is of degree $\le s_1 + s_2 + \cdots + s_n$, where $s_1, s_2, \ldots, s_n$ are nonnegative integers, then*

$$\left(\frac{\partial}{\partial x_1}\right)^{s_1}\left(\frac{\partial}{\partial x_2}\right)^{s_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{s_n} P(x_1, x_2, \ldots, x_n)$$
$$= \sum_{a_1=0}^{s_1}\cdots\sum_{a_n=0}^{s_n}(-1)^{s_1+a_1}\binom{s_1}{a_1}\cdots(-1)^{s_n+a_n}\binom{s_n}{a_n}P(a_1, a_2, \ldots, a_n).$$

It turns out that it is usually difficult to apply this lemma directly to $P(G)$. Instead, we shall introduce another polynomial $Q(G)$ so that for the mappings φ of our interest, $[P(G)]_\varphi = [Q(G)]_\varphi$. Then we apply Lemma 3 to polynomial $Q(G)$.

Suppose $Q(G)$ is a polynomial with variables $\{x_v : v \in V(G)\}$. For a mapping $\sigma : V(G) \to \mathbb{Z}$, let $Q(G, \sigma)$ be the evaluation of $Q(G)$ at $x_v = \sigma(v)$. The following corollary follows easily from Lemma 3.

Corollary 4. *Let $\varphi : V(G) \to \{0, 1, \ldots\}$ be a mapping with $\sum_{v\in V(G)} \varphi(v) = \deg Q(G)$. Then*

$$[Q(G)]_\varphi \prod_{v\in V(G)} \varphi(v)! = \sum_\sigma \left((-1)^{\sum_{v\in V(G)}(\varphi(v)+\sigma(v))} \prod_{v\in V(G)} \binom{\varphi(v)}{\sigma(v)}\right) Q(G, \sigma),$$

where the summation is over all the mapping $\sigma : V(G) \to \mathbb{Z}$ with $0 \le \sigma(v) \le \varphi(v)$ for all $v \in V(G)$.

3. A warm up

First we prove that if $G = K_{2*k}$, then G is on-line k-choosable. This is a special case of the results to be proved later. We prove it separately to exhibit the idea used in the proof.

For convenience, we denote the vertices of G by $v_{i,j}$, $j = 1, 2$, $i = 1, 2, \ldots, k$ and the ith partite set is $\{v_{i,1}, v_{i,2}\}$. Order the vertices by the lexicographic order of their indices, i.e., $v_{i,j} < v_{i',j'}$ means either $i < i'$ or $i = i'$ and $j < j'$. It was proved in [3] that K_{2*k} is k-choosable. The proof is easy: Assume L is a list assignment with $|L(v)| = n$ for all $v \in V(G)$. If for some i, $L(v_{i,1}) \cap L(v_{i,2}) \ne \emptyset$, then colour $v_{i,1}$ and $v_{i,2}$ with a common colour, and colour the remaining part of G by induction. If $L(v_{i,1}) \cap L(v_{i,2}) = \emptyset$ for all i, then by the well-known Hall's Theorem for the existence of distinct representatives of a set system, one can find a distinct permissible colour for each vertex $v_{i,j}$ in $L(v_{i,j})$.

Theorem 5. *For any positive integer k, $G = K_{2*k}$ is on-line k-choosable.*

As mentioned above, the proof above for the k-choosability of G does not work for the on-line version. We shall present two proofs for Theorem 5. Both proofs are to show that the coefficient of a certain monomial in a polynomial is nonzero. The first proof is a standard calculation, and the second proof uses an idea that will be useful for the proof of the more general cases.

We shall prove that for the mapping φ defined as $\varphi(v) = k - 1$ for all $v \in V(G)$, $[P(G)]_\varphi \ne 0$. This would imply that G is on-line k-choosable.

First proof

By Corollary 4, with $\varphi(v) = k - 1$ for all $v \in V(G)$, we have

$$[P(G)]_\varphi((k-1)!)^{2k} = \sum_\sigma \left((-1)^{\sum_{v\in V(G)}\sigma(v)} \prod_{v\in V(G)} \binom{k-1}{\sigma(v)}\right) P(G, \sigma),$$

where the last sum runs over all the mappings $\sigma : V(G) \to \{0, 1, 2, \ldots, k-1\}$. It follows from the definition of $P(G)$ that if $\sigma(v_{i,j}) = \sigma(v_{i',j'})$ for some $i \ne i'$, then $P(G, \sigma) = 0$. So the sum can be taken to run over all mappings $\sigma : V(G) \to \{0, 1, 2, \ldots, k-1\}$ such that for any $i \ne i'$, $\sigma(v_{i,j}) \ne \sigma(v_{i',j'})$.

Let Ω be the set of all one-to-one mappings $\tau : \{1, 2, \ldots, k\} \to \{0, 1, \ldots, k-1\}$. For $\tau \in \Omega$, let $\sigma_\tau : V(G) \to \{0, 1, 2, \ldots, k-1\}$ be the mapping defined as $\sigma_\tau(v_{i,j}) = \tau(i)$. It is straightforward to verify that $\sigma(v_{i,j}) \neq \sigma(v_{i',j'})$ for all $i \neq i'$ if and only if $\sigma = \sigma_\tau$ for some $\tau \in \Omega$.

Given $\tau \in \Omega$, the multi-set $\{\sigma_\tau(v) : v \in V(G)\}$ is equal to $\{0, 0, 1, 1, 2, 2, \ldots, k-1, k-1\}$. Hence

$$\prod_{v \in V(G)} \binom{k-1}{\sigma_\tau(v)} = \prod_{i=0}^{k-1} \binom{k-1}{i}^2.$$

$$(-1)^{\sum_{v \in V(G)} \sigma_\tau(v)} = 1.$$

$$\begin{aligned} P(G, \sigma_\tau) &= \prod_{0 \le i < i' \le k-1} (i - i')^4 \\ &= \prod_{i=1}^{k-1} (i!)^4 \qquad . \end{aligned}$$

So

$$\begin{aligned} [P(G)]_\varphi ((k-1)!)^{2k} &= \sum_{\tau \in \Omega} \left((-1)^{\sum_{v \in V(G)} \sigma_\tau(v)} \prod_{v \in V(G)} \binom{k-1}{\sigma_\tau(v)} \right) P(G, \sigma_\tau) \\ &= |\Omega| \left(\prod_{i=0}^{k-1} \binom{k-1}{i}^2 (i!(k-1-i)!)^2 \right) \\ &= k!((k-1)!)^{2k}. \end{aligned}$$

It follows that $[P(G)]_\varphi = k! \neq 0$. Hence K_{2*k} is on-line k-choosable.

Second proof

Let E_1 be the set of edges with at least one end vertex in $\{v_{i,1} : i = 1, 2, \ldots, k\}$, and let $E_2 = E(G) \backslash E_1$ which is the set of edges with both ends in $\{v_{i,2} : i = 1, 2, \ldots, k\}$. Note that the subgraph of G induced by E_1 is a spanning subgraph of G which is uniquely k-colourable.

We consider a different polynomial:

$$Q(G) = \prod_{u<v, uv \in E_1} (x_u - x_v) \prod_{u<v, uv \in E_2} (x_u - x_v - 1).$$

This polynomial seems have nothing to do with colouring of G. However, for $\varphi : V(G) \to \{0, 1, \ldots\}$ with $\sum_{v \in V(G)} \varphi(v) = |E(G)|$,

$$[Q(G)]_\varphi = [P(G)]_\varphi.$$

Applying Lemma 3 to the polynomial $Q(G)$ with $\varphi(v) = k-1$ for all v, we conclude that

$$[Q(G)]_\varphi ((k-1)!)^{2k} = \sum_\sigma \left((-1)^{\sum_{v \in V(G)} \sigma(v)} \prod_{v \in V(G)} \binom{k-1}{\sigma(v)} \right) Q(G, \sigma),$$

where the last sum runs over all the mappings $\sigma : V(G) \to \{0, 1, 2, \ldots, k-1\}$. Obviously, it suffices to restrict the summation to those σ for which $Q(G, \sigma) \neq 0$, i.e., those σ such that $\sigma(u) \neq \sigma(v)$ if $uv \in E_1$, and $\sigma(u) \neq \sigma(v) + 1$ if $uv \in E_2$ and $u < v$. Such a σ is a proper colouring of $G_1 = (V, E_1)$. As G_1 is uniquely k-colourable, we conclude that $\sigma(v_{i,j}) = \sigma(v_{i',j'})$ if and only if $i = i'$. Moreover, for any $i < i'$, $\sigma(v_{i,j}) \neq \sigma(v_{i',j'}) + 1$. The only σ satisfying these requirements is that $\sigma(v_{i,j}) = i-1$ for all i. So the summation above has exactly one nonzero entry, and hence the sum itself is nonzero. Therefore $[Q(G)]_\varphi = [P(G)]_\varphi \neq 0$ and G is on-line k-choosable.

Argument for other complete k-partite graphs

In later sections, we prove that some other complete k-partite graphs are on-line k-choosable. The proofs are similar to the second proof above. Assume G is a complete k-partite graph with partite sets $V_i = \{v_{i,j} : j = 1, 2, \dots, \ell_i\}$ $(i = 1, 2, \dots, k)$. Let E_1 be the set of edges of G with at least one end vertex in $\{v_{i,1} : i = 1, 2, \dots, k\}$. We consider the following polynomial:

$$Q(G) = \prod_{u<v, uv \in E_1} (x_u - x_v) \prod_{u<v, e=uv \in E \setminus E_1} (x_u - x_v - \psi(e)), \tag{1}$$

where $\psi(e) \in \mathbb{Z}$ will be defined later.

Assume $\varphi : V(G) \to \{0, 1, \dots, k-1\}$ is given such that $\sum_{v \in V(G)} \varphi(v) = |E(G)| = \deg(Q(G))$. We would like to prove that the coefficient $[Q(G)]_\varphi$ is nonzero. Applying Lemma 3 to the polynomial $Q(G)$ and φ, we know that $[Q(G)]_\varphi \neq 0$ if and only if

$$\sum_{\sigma} \left((-1)^{\sum_{v \in V(G)} (\varphi(v) + \sigma(v))} \prod_{v \in V(G)} \binom{\varphi(v)}{\sigma(v)} \right) Q(G, \sigma)$$

is non-zero. The summation is over all mappings $\sigma : V \to \{0, 1, \dots, k-1\}$ for which $\sigma(v) \leq \varphi(v)$ and $Q(G, \sigma) \neq 0$.

The key idea is to choose the mapping ψ in (1) so that there are very few mappings σ for which $Q(G, \sigma) \neq 0$.

The spanning subgraph of G induced by E_1 is uniquely k-colourable. By the same argument as in the second proof above, we conclude that for any mapping $\sigma : V \to \{0, 1, \dots, k-1\}$ for which $Q(G, \sigma) \neq 0$, $\sigma(v_{i,j}) = \sigma(v_{i',j'})$ if and only if $i = i'$. Let $\sigma_i = \sigma(V_i)$. We call σ_i the *colour* of V_i. Thus the sequence $(\sigma_1, \sigma_2, \dots, \sigma_k)$ is a permutation of $\{0, 1, \dots, k-1\}$.

Just like the second proof of Theorem 5 presented above, for edges $e \in E \setminus E_1$, the numbers $\psi(e)$ will be chosen to put further restrictions on the 'allowable' permutations. If there is an edge $e = uv$ connecting a vertex $u \in V_i$ and a vertex $v \in V_j$ and $u < v$, then $(\sigma_i - \sigma_j - \psi(e))$ is one factor of $Q(G, \sigma)$. As $Q(G, \sigma) \neq 0$, the difference $\sigma_i - \sigma_j$ cannot be $\psi(e)$. So each edge $e \notin E_1$ between V_i and V_j put one restriction on the difference $\sigma_i - \sigma_j$.

In some cases, by choosing the functions φ and ψ appropriately, there is a unique σ for which $Q(G, \sigma) \neq 0$. In this case, it follows that $[Q(G)]_\varphi \neq 0$. In other cases, we find functions φ and ψ to reduce the number of σ's for which $Q(G, \sigma) \neq 0$, so that the calculation of $[Q(G)]_\varphi$ becomes feasible.

4. $K_{2*(k-\ell),\ell+1,1*(\ell-1)}$

Theorem 6. *For any positive integers $1 \leq \ell \leq k$, the complete partite graph $K_{2*(k-\ell),\ell+1,1*(\ell-1)}$ is on-line k-choosable.*

Proof. Note that if $\ell = 1$, then the graph under consideration is K_{2*k}. Let $V_1, V_2, \dots, V_k$ be the partite sets with $|V_1| = |V_2| = \cdots = |V_{k-\ell}| = 2$, $|V_{k-\ell+1}| = \ell + 1$ and $|V_{k-\ell+2}| = \cdots = |V_k| = 1$.

Let $\varphi : V(G) \to \{0, 1, \dots\}$ be defined as

$$\varphi(v) = \begin{cases} k-1, & \text{if } v \in V_i \text{ for } i = 1, 2, \dots, k-\ell+1 \text{ and } v \neq v_{k-\ell+1,\ell+1} \\ k-\ell, & \text{if } v = v_{k-\ell+1,\ell+1}. \\ i-1, & \text{if } v = v_{i,1} \text{ for } i = k-\ell+2, \dots, k. \end{cases}$$

Now we follow the argument at the end of Section 3. Let σ be a mapping for which $0 \leq \sigma(v) \leq \varphi(v)$ and $Q(G, \sigma) \neq 0$. By the argument in Section 3, there is a permutation $(\sigma_1, \sigma_2, \dots, \sigma_k)$ of $\{0, 1, \dots, k-1\}$ such that $\sigma(V_i) = \sigma_i$.

For $1 \leq i < i' \leq k - \ell$, there is one edge $e \notin E_1$ between E_i and $E_{i'}$. So we can put one restriction on the difference $\sigma_i - \sigma_{i'}$. For $1 \leq i \leq k - \ell$, there are ℓ edges $e \notin E_1$ between V_i and $V_{k-\ell+1}$. So we can put ℓ restrictions on the difference $\sigma_i - \sigma_{k-\ell+1}$. We choose these restrictions so that

- $\sigma_i \neq \sigma_{i'} + 1$ if $1 \leq i < i' \leq k - \ell$.
- For $i = 1, 2, \dots, k - \ell$, $\sigma_i \neq \sigma_{k-\ell+1} + s$, $s \in \{1, \dots, \ell\}$.

As $\sigma_1, \sigma_2, \ldots, \sigma_{k-\ell}$ are $k-\ell$ distinct elements in the set $\{0, 1, \ldots, k-1\} \setminus \{\sigma_{k-\ell+1} + s : s = 0, 1, \ldots, \ell\}$, we must have

$$|\{0, 1, \ldots, k-1\} \setminus \{\sigma_{k-\ell+1} + s : s = 0, 1, \ldots, \ell\}| \geq k - \ell.$$

As $\{\sigma_{k-\ell+1} + s : s = 0, 1, \ldots, \ell\}$ has cardinality $\ell + 1$, it cannot be a subset of $\{0, 1, \ldots, k-1\}$. Since $\sigma_{k-\ell+1} \geq 0$, this implies that $\sigma_{k-\ell+1} \geq k-\ell$. On the other hand, $\sigma_{k-\ell+1} \leq \varphi(v_{k-\ell+1,\ell+1}) = k-\ell$. Therefore $\sigma_{k-\ell+1} = k-\ell$, and $\{\sigma_1, \sigma_2, \ldots, \sigma_{k-\ell}\} = \{0, 1, \ldots, k-\ell-1\}$. Using the restrictions that $\sigma_i \neq \sigma_{i'} + 1$ if $1 \leq i < i' \leq k-\ell$, we conclude that

$$\sigma_i = i - 1, \quad \text{for } i = 1, 2, \ldots, k-\ell.$$

For $i = k-\ell+2, \ldots, k$, since $\sigma_i \notin \{0, 1, \ldots, k-\ell\}$ and $\sigma_i \leq \varphi(v_{1,i}) = i-1$, we conclude that

$$\sigma_i = i - 1, \quad \text{for } i = k-\ell+2, \ldots, k.$$

So there is a unique σ, i.e., $\sigma(v_{i,j}) = i - 1$ for $i = 1, 2, \ldots, k$, such that $Q(G, \sigma) \neq 0$. Therefore

$$[Q(G)]_\varphi \prod_{v \in V(G)} (\varphi(v))! = \sum_\sigma \left((-1)^{\sum_{v\in V(G)}(\varphi(v)+\sigma(v))} \prod_{v \in V(G)} \binom{\varphi(v)}{\sigma(v)} \right) Q(G, \sigma) \neq 0. \quad \square$$

5. Limitation of the polynomial method

One might hope that the polynomial method used in the previous sections can be applied to all other cases of Conjecture 4. However, it is easy to see that there are some cases that cannot be solved by this method. The following lemma implies that for some cases of Conjecture 4, the polynomial method used in the previous sections does not work.

Lemma 7. *Suppose G is a complete multipartite graph. Assume there are two partite sets $V_1 = \{v_1, v_2, \ldots, v_{2t+1}\}$ and $V_2 = \{u_1, u_2, \ldots, u_{2t+1}\}$ of the same odd cardinality. For any $\varphi : V(G) \to \{0, 1, \ldots, \}$ with $\varphi(u_j) = \varphi(v_j)$ for $j = 1, 2, \ldots, 2t+1$, $[P(G)]_\varphi = 0$.*

Proof. Let $P'(G)$ be obtained from $P(G)$ by interchanging x_{u_j} and x_{v_j} for $j = 1, 2, \ldots, 2t+1$. For every vertex $v \notin V_1 \cup V_2$, the two edges $u_j v$ and $v_j v$ contribute the same product (the same sign) in $P'(G)$ and $P(G)$. For each edge $u_j v_{j'}$ connecting a vertex of V_1 and V_2, its contributions to $P(G)$ and $P'(G)$ differ by a sign. Therefore $P'(G) = -P(G)$. On the other hand, as $\varphi(u_j) = \varphi(v_j)$ for $j = 1, 2, \ldots, 2t+1$, we have $[P(G)]_\varphi = [P'(G)]_\varphi$. Therefore $[P(G)]_\varphi = 0$. $\square$

Corollary 8. *Assume $m \geq 3$ and $k \geq 2m$ and $G = K_{3*m,1*m,2*(k-2m)}$. For any $\varphi : V(G) \to \{0, 1, \ldots, k-1\}$, $[P(G)]_\varphi = 0$.*

Proof. Observe that $|V(G)| = 2k$ and $|E(G)| = 2k(k-1) - m$. If $\varphi : V(G) \to \{0, 1, \ldots, k-1\}$ satisfies $\sum_{v\in V(G)} \varphi(v) = |E(G)| = \deg(P(G))$, there are at most m vertices v for which $\varphi(v) \neq k-1$. As $m \geq 3$, there are two partite sets both of cardinality 3 or both of cardinality 1 such that for any vertex v of these two partite sets, $\varphi(v) = k-1$. By Lemma 7, $[P(G)]_\varphi = 0$. $\square$

Corollary 9. *Assume $s \geq 4$ and $k \geq 7$ and $G = K_{s,k+2-s,1*(k-2)}$. For any $\varphi : V(G) \to \{0, 1, \ldots, k-1\}$, $[P(G)]_\varphi = 0$.*

The proof of Corollary 9 is the same as that of Corollary 8, and is omitted.

6. $K_{s,t,1*(k-2)}$ and $K_{3*2,1*2,2*(k-4)}$

By Corollary 9, if $s, t \geq 5$ and $G = K_{s,t,1*(k-2)}$ has $2k$ vertices, then G cannot be proven to be chromatic-choosable by the polynomial method used in previous sections, although Conjecture 4 asserts the graph is on-line chromatic choosable. However, if $|V(G)|$ is much less than $2k$, then it is still possible to apply the polynomial method to prove that G is on-line chromatic-choosable.

Theorem 10. *Assume that s, t, k are positive integers such that $(s-1)(t-1) \leq 2k-3$. Then $K_{s,t,1*(k-2)}$ is on-line k-choosable.*

Proof. To prove this theorem, we only need to consider the cases that $(s-1)(t-1) = 2k-3$ and $(s-1)(t-1) = 2k-4$. If $(s-1)(t-1) < 2k-4$, then we can delete a singleton partite set (so k is decreased by 1, and every vertex has one less permissible colour. If such a graph is on-line chromatic-choosable, then by adding back the singleton partite set, the graph is still on-line chromatic-choosable).

Let $G = K_{s,t,1*(k-2)}$. Let $V_1, V_2, \ldots, V_k$ be the partite sets with $|V_1| = s$, $|V_2| = t$ and $|V_3| = \cdots = |V_k| = 1$. We follow the argument at the end of Section 3.

If $(s-1)(t-1) = 2k-4$, then let $\varphi : V(G) \to \{0, 1, \ldots\}$ be defined as

$$\varphi(v) = \begin{cases} k-1, & \text{if } v \in V_1 \cup V_2 \text{ and } v \neq v_{1,s} \\ k-2, & \text{if } v = v_{1,s}. \\ i-1, & \text{if } v = v_{i,1} \text{ for } i = 3, 4, \ldots, k. \end{cases}$$

If $(s-1)(t-1) = 2k-3$, then the mapping φ is slightly different.

$$\varphi(v) = \begin{cases} k-1, & \text{if } v \in V_1 \cup V_2 \\ i-1, & \text{if } v = v_{i,1} \text{ for } i = 3, 4, \ldots, k. \end{cases}$$

Let σ be a mapping for which $0 \leq \sigma(v) \leq \varphi(v)$ and $Q(G, \sigma) \neq 0$. By the argument in Section 3, there is a permutation $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ of $\{0, 1, \ldots, k-1\}$ such that $\sigma(v_{i,j}) = \sigma_i$.

If $(s-1)(t-1) = 2k-4$, then we can put $2k-4$ restrictions on the difference $\sigma_1 - \sigma_2$. We choose these restrictions so that $\sigma_1 - \sigma_2 \neq \pm 1, \pm 2, \ldots, \pm(k-2)$. If $(s-1)(t-1) = 2k-3$, then we can put $2k-3$ restrictions on the difference $\sigma_1 - \sigma_2$ and we choose the restrictions so that $\sigma_1 - \sigma_2 \neq \pm 1, \pm 2, \ldots, \pm(k-2), (k-1)$.

As $\sigma_1, \sigma_2 \in \{0, 1, \ldots, k-1\}$, we conclude that $|\sigma_1 - \sigma_2| = k-1$. When $(s-1)(t-1) = 2k-4$, $\sigma_1 \leq \varphi(v_{1,s}) = k-2$. When $(s-1)(t-1) = 2k-3$, $\sigma_1 - \sigma_2 \neq k-1$. So the only possibility is that

$$\sigma_1 = 0, \qquad \sigma_2 = k-1.$$

Hence $(\sigma_3, \sigma_4, \ldots, \sigma_k)$ is a permutation of $\{1, 2, \ldots, k-2\}$. Moreover, $\sigma_i \leq \varphi(v_{i,1}) = i-1$ for $i = 3, 4, \ldots, k-1$.

Let S_{k-2} be the set of all permutations of $\{1, 2, \ldots, k-2\}$. For $\tau \in S_{k-2}$, let σ_τ be the mapping such that the corresponding σ_i satisfies $\sigma_1 = 0$, $\sigma_2 = k-1$ and $\sigma_{i+2} = \tau(i)$. Let $\Omega_{k-2} \subseteq S_{k-2}$ be the set of all permutations τ of $\{1, 2, \ldots, k-2\}$ satisfying $\tau(i) \in \{1, 2, \ldots, i+1\}$ for $1 \leq i \leq k-2$.

Lemma 11. *For $k \geq 3$, let Ω_{k-2} be the set of permutations of $\{1, 2, \ldots, k-2\}$ such that $\tau(i) \in \{1, 2, \ldots, i+1\}$. Then*

$$\sum_{\tau \in \Omega_{k-2}} \mathrm{sign}(\tau) \prod_{i=1}^{k-2} \binom{i+1}{\tau(i)} \neq 0.$$

We shall prove Lemma 11 later. Now we use this lemma to continue our proof of Theorem 10.

By the argument above, we have

$$[Q(G)]_\varphi \left(\prod_{v \in V(G)} (\varphi(v))! \right) = \sum_{\tau \in \Omega_{k-2}} \left((-1)^{\sum_{v \in V(G)} (\varphi(v) + \sigma_\tau(v))} \prod_{v \in V(G)} \binom{\varphi(v)}{\sigma_\tau(v)} \right) Q(G, \sigma_\tau).$$

Note that for $\tau \in \Omega_{k-2}$, $\sum_{v\in V(G)}(\varphi(v)+\sigma_{\tau'}(v))$ is a constant. If $\tau \in \Omega_{k-2}$ and τ' is obtained from τ by composing with a convolution, i.e., interchanging two entries, then

$$Q(G,\sigma_{\tau'}) = -Q(G,\sigma_\tau).$$

Therefore for $\tau \in \Omega_{k-2}$,

$$Q(G,\sigma_\tau) = \operatorname{sign}(\tau)C$$

for some nonzero constant C. Moreover, for $\tau \in \Omega_{k-2}$,

$$\prod_{v\in V(G)}\binom{\varphi(v)}{\sigma_\tau(v)} = \prod_{i=1}^{k-2}\binom{i+1}{\tau(i)}.$$

By Lemma 11,

$$\sum_{\tau\in\Omega_{k-2}}\operatorname{sign}(\tau)\prod_{v\in V(G)}\binom{\varphi(v)}{\sigma_\tau(v)} = \sum_{\tau\in\Omega_{k-2}}\operatorname{sign}(\tau)\prod_{i=1}^{k-2}\binom{i+1}{\tau(i)} \neq 0.$$

Hence $[Q(G)]_\varphi \neq 0$. $\square$

We still need to prove Lemma 11. Instead of proving Lemma 11, we prove a more general result.

Lemma 12. *For $k \geq 3, \ell \geq 1$, let $\Omega_{k-2,\ell}$ be the set of permutations of $\{1,2,\ldots,k-2\}$ such that $\tau(i) \in \{1,2,\ldots,i+\ell\}$. Then*

$$\sum_{\tau\in\Omega_{k-2,\ell}}\operatorname{sign}(\tau)\prod_{i=1}^{k-2}\binom{i+\ell}{\tau(i)} \neq 0.$$

Proof. The proof uses Lemma 3, however, in the opposite direction. Instead of using the formula in Lemma 3 to prove that the coefficient of a monomial is non-zero, we use the fact that the coefficient of a monomial is non-zero to show that a certain summation is non-zero.

Consider the polynomial $P(x_1,x_2,\ldots,x_{k-2})$ defined as

$$P(x_1,x_2,\ldots,x_{k-2}) = \prod_{1\leq i<j\leq k-2}(x_i-x_j)\prod_{i=1}^{k-2}\left(x_i\prod_{j=k-1}^{k-2+\ell}(x_i-j)\right).$$

Let α be the coefficient of the monomial $\prod_{i=1}^{k-2}x_i^{i+\ell}$ in the polynomial $P(x_1,x_2,\ldots,x_{k-2})$. In expanding the product in P to obtain the monomial $\prod_{i=1}^{k-2}x_i^{i+\ell}$, we need to choose x_{k-2} whenever possible, for otherwise, the degree of x_{k-2} cannot be $k-2+\ell$, subject to this, we choose x_{k-3} whenever possible, and so on. So $\alpha = 1$ or -1.

On the other hand, by Lemma 3,

$$\alpha\cdot\prod_{i=1}^{k-1}(i+\ell)! = \sum_\sigma(-1)^{\sum_{i=1}^{k-2}(i+\ell+\sigma(i))}\prod_{i=1}^{k-2}\binom{i+\ell}{\sigma(i)}P(\sigma(1),\sigma(2),\ldots,\sigma(k-2)),$$

where the sum runs over all the mappings $\sigma : \{1,2,\ldots,k-2\} \to \mathbb{Z}$ satisfies $0 \leq \sigma(i) \leq i+\ell$ for $i = 1,2,\ldots,k-2$. Of course we may restrict to those σ for which $P(\sigma(1),\sigma(2),\ldots,\sigma(k-2)) \neq 0$. It is easy to see that such a σ is a permutation of $\{1,2,\ldots,k-2\}$ with $\sigma(i) \leq i+\ell$. In other words, $\sigma \in \Omega_{k-2,\ell}$. As σ is a permutation of $\{1,2,\ldots,k-2\}$,

$$(-1)^{\sum_{i=1}^{k-2}(i+\ell+\sigma(i))}P(\sigma(1),\sigma(2),\ldots,\sigma(k-2)) = \operatorname{sign}(\sigma)C$$

for some nonzero constant C.

Therefore,

$$\alpha \cdot \prod_{i=1}^{k-1}(i+\ell)! = \left(\sum_{\sigma \in \Omega_{k-2,\ell}} \operatorname{sign}(\sigma) \prod_{i=1}^{k-2} \binom{i+\ell}{\sigma(i)} \right) C.$$

As $\alpha \neq 0$, we conclude that $\sum_{\sigma \in \Omega_{k-2,\ell}} \operatorname{sign}(\sigma) \prod_{i=1}^{k-2} \binom{i+\ell}{\sigma(i)} \neq 0$. $\square$

Note that in Theorem 10, if s, t are both big, say close to $1+\sqrt{2k-3}$, then the number of vertices in $K_{s,t,1*(k-2)}$ is close to $k+2\sqrt{2k}$. However, for $s=2$, $t=2k-2$, $K_{s,t,1*(k-2)}$ has $3k-2$ vertices, and for $s=3$ and $t=k-1$, $K_{s,t,1*(k-2)}$ has $2k$ vertices.

Theorem 13. *For $k \geq 4$, $K_{3*2,1*2,2*(k-4)}$ are on-line k-choosable.*

Proof. Let $V_1, V_2, \ldots, V_k$ be the partite sets with $|V_1| = |V_2| = 3$, $|V_3| = |V_4| = 1$ and $|V_5| = \cdots = |V_k| = 2$. Let $\varphi : V(G) \to \{0, 1, \ldots\}$ be defined as

$$\varphi(v) = \begin{cases} k-2, & \text{if } v = v_{3,1} \text{ or } v = v_{1,3} \\ k-1, & \text{otherwise.} \end{cases}$$

Let σ be a mapping for which $0 \leq \sigma(v) \leq \varphi(v)$ and $Q(G, \sigma) \neq 0$. By the argument in Section 3, there is a permutation $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ of $\{0, 1, \ldots, k-1\}$ such that $\sigma(v_{i,j}) = \sigma_i$.

We follow the argument at the end of Section 3. We can put 4 restrictions on the difference $\sigma_1 - \sigma_2$, and 2 restrictions on the difference $\sigma_i - \sigma_j$ for $i = 1, 2$ and $j = 5, 6, \ldots, k$. We choose these restrictions as

- $\sigma_1 \neq \sigma_2 \pm 1, \sigma_2 \pm 2$.
- $\sigma_i \neq \sigma_j \pm 1$ for $i = 1, 2$ and $j = 5, 6, \ldots, k$.

First we show that if $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ of $\{0, 1, \ldots, k-1\}$ satisfying the restrictions above, then $\sigma_1 = 0$.

Assume to the contrary that $\sigma_1 \neq 0$. Since $\sigma_1 \leq \varphi(v_{1,3}) = k-2$, we conclude that $\sigma_1 \pm 1 \in \{0, 1, \ldots, k-1\}$. As $\sigma_1 \pm 1 \neq \sigma_j$ for $j = 2, 5, 6, \ldots, k$, we conclude that $\{\sigma_3, \sigma_4\} = \{\sigma_1 - 1, \sigma_1 + 1\}$. As $\sigma_2 \neq \sigma_1 \pm 2$, it follows that $\{\sigma_3, \sigma_4\} \cap \{\sigma_2 - 1, \sigma_2 + 1\} = \emptyset$. By the restrictions above, $\{\sigma_5, \sigma_6, \ldots, \sigma_k\} \cap \{\sigma_2 - 1, \sigma_2 + 1\} = \emptyset$. So $\{\sigma_2 - 1, \sigma_2 + 1\} \cap \{\sigma_1, \sigma_2, \ldots, \sigma_k\} = \emptyset$. This is a contradiction, as at least one of $\sigma_2 - 1, \sigma_2 + 1$ lies in $\{0, 1, \ldots, k-1\}$.

Thus we have proved that $\sigma_1 = 0$. This forces $\sigma_3 = 1$ or $\sigma_4 = 1$. Since we cannot have $\{\sigma_3, \sigma_4\} = \{\sigma_2 - 1, \sigma_2 + 1\}$, we must have $\sigma_2 = k-1$, and $\sigma_4 = k-2$ or $\sigma_3 = k-2$.

We put one restriction on $\sigma_i - \sigma_j$ as $\sigma_i - \sigma_j \neq 1$ for $5 \leq i < j \leq k$. This forces $\sigma_i = i - 3$ for $i = 5, 6, \ldots, k$. There are two choices of $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ for which $Q(G, \sigma) \neq 0$. It is easy to see that the summation of these two entries is equal to $\left(\binom{k-2}{k-2}\binom{k-1}{1} - \binom{k-1}{k-2}\binom{k-2}{1} \right) C$ for some nonzero constant C. Hence $[Q(G)]_\varphi \neq 0$ and G is on-line k-choosable. $\square$

7. $K_{4,3,1*3,2*(k-5)}$

Theorem 14. *For $k \geq 5$, $K_{4,3,1*3,2*(k-5)}$ is on-line k-choosable.*

Proof. This graph is only slightly different from the graph studied in Theorem 13. However, the calculation is more complicated and we shall introduce a new idea to simplify the calculation.

Let $V_1, V_2, \ldots, V_k$ be the partite sets with $|V_1| = 4$, $|V_2| = 3$, $|V_3| = |V_4| = |V_5| = 1$ and $|V_6| = \cdots = |V_k| = 2$. Let $\varphi : V(G) \to \{0, 1, \ldots, k-1\}$ be defined as

$$\varphi(v) = \begin{cases} k-2, & \text{if } v = v_{4,1} \text{ or } v = v_{1,4} \\ k-3, & \text{if } v = v_{3,1} \\ k-1, & \text{otherwise.} \end{cases}$$

Let $Q(G)$ be the polynomial defined as in the discussion at the end of Section 3. (In the definition of $Q(G)$, those constants $\psi(e)$ for $e \notin E_1$ still need to be determined later.) We need to show that $[Q(G)]_\varphi \neq 0$.

Instead of applying Lemma 3 to the polynomial $Q(G)$, we consider the following polynomial

$$R(G) = Q(G) \cdot (x_{v_{1,4}} - (k-1))(x_{v_{4,1}} - x_{v_{1,1}} - 1)(x_{v_{3,1}} - x_{v_{1,1}} - 1)(x_{v_{3,1}} - x_{v_{6,1}} + 1).$$

Let φ' be the constant mapping $\varphi'(v) = k-1$ for all $v \in V(G)$.

Claim. $[Q(G)]_{\varphi} = [R(G)]_{\varphi'}$.

Let $S(G) = (x_{v_{1,4}} - (k-1))(x_{v_{4,1}} - x_{v_{1,1}} - 1)(x_{v_{3,1}} - x_{v_{1,1}} - 1)(x_{v_{3,1}} - x_{v_{6,1}} + 1)$. The degree of $S(G)$ is 4. For any mapping $\xi : V(G) \to \mathbb{N}$, let $\bar{\xi}(v) = k - 1 - \xi(v)$. We have

$$[R(G)]_{\varphi'} = \sum_{\xi} [Q(G)]_{\bar{\xi}} [S(G)]_{\xi}$$

where ξ runs over all the mappings $\xi : V(G) \to \mathbb{N}$. Of course, we only need to consider those ξ for which $[Q(G)]_{\bar{\xi}}[S(G)]_{\xi} \neq 0$. For such ξ, we must have $\sum_{v \in V(G)} \xi(v) = 4$, $\xi(v_{1,4}) = 1$, $\xi(v_{4,1}) = 0$ or 1, $\xi(v_{3,1}) = 0$, 1, or 2.

If $\xi(v_{4,1}) = 0$ (respectively, $\xi(v_{3,1}) = 0$), then $\bar{\xi}(v_{4,1}) = \bar{\xi}(v_{5,1}) = k-1$ (respectively, $\bar{\xi}(v_{3,1}) = \bar{\xi}(v_{5,1}) = k-1$). It follows from Lemma 7 that $[Q(G)]_{\bar{\xi}} = 0$. Thus we may assume that $\xi(v_{4,1}) = 1$ and $\xi(v_{3,1}) = 1$ or 2. If $\xi(v_{3,1}) = 1$, then $\bar{\xi}(v_{3,1}) = \bar{\xi}(v_{4,1}) = k-2$, again it follows from Lemma 7 that $[Q(G)]_{\bar{\xi}} = 0$. Therefore, we may assume that $\xi(v_{3,1}) = 2$. This implies that $\xi(v) = 0$ for all other vertices v, and $[S(G)]_{\xi} = 1$. For such a mapping ξ, we have $\bar{\xi} = \varphi$. It follows that $[Q(G)]_{\varphi} = [R(G)]_{\varphi'}$.

Now we apply Lemma 3 to calculate $[R(G)]_{\varphi'}$. Similarly as the argument at the end of Section 3, $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ is a permutation of $\{0, 1, \ldots, k-1\}$ as described in the general argument.

By considering edges not in E_1, we can put six restriction on $\sigma_1 - \sigma_2$, one restriction on the difference $\sigma_i - \sigma_j$ for $6 \leq i < j \leq k$, two restrictions on the difference $\sigma_2 - \sigma_i$, and three restrictions on $\sigma_1 - \sigma_i$ for $6 \leq i \leq k$. We choose these restrictions as follows:

- $\sigma_1 - \sigma_2 \neq \pm 1, \pm 2, \pm 3$.
- $\sigma_i - \sigma_j \neq 1$ for $6 \leq i < j \leq k$.
- $\sigma_2 - \sigma_i \neq \pm 1$ for $6 \leq i \leq k$.
- $\sigma_1 - \sigma_i \neq \pm 1, -2$ for $6 \leq i \leq k$.

For $R(G, \sigma) \neq 0$, we have the following restrictions:

- $\sigma_3 - \sigma_1 \neq 1$.
- $\sigma_6 - \sigma_3 \neq 1$.
- $\sigma_4 - \sigma_1 \neq 1$.
- $\sigma_1 \neq k - 1$.

We consider two cases.

Case 1 $\sigma_1 = 0$. The restrictions above show that $\sigma_i \neq 1$ for $i \neq 5$. Hence $\sigma_5 = 1$. Also $\sigma_i \neq 2$ for $i \neq 3, 4$. So either $\sigma_3 = 2$ or $\sigma_4 = 2$. If $\sigma_2 \neq k-1$, then $\sigma_2 \pm 1 \in \{2, 3, \ldots, k-1\}$. The restrictions above show that $\{\sigma_2 + 1, \sigma_2 - 1\} \subseteq \{\sigma_3, \sigma_4\}$. This forces $\sigma_2 - \sigma_1 = 3$, in contradiction to the restriction above. Hence we must have $\sigma_2 = k-1$, and $\{\sigma_3, \sigma_4\} = \{2, k-2\}$. As $\sigma_i - \sigma_j \neq 1$ for $6 \leq i < j \leq k$, we conclude that for $i = 6, 7, \ldots, k$, $\sigma_i = i - 3$. As $\sigma_6 - 1 \neq \sigma_3$, we conclude that $\sigma_3 = k-2$ and $\sigma_4 = 2$.

Case 2 $\sigma_1 \neq 0$. Then $\sigma_1 \pm 1 \in \{0, 1, \ldots, k-1\}$. The above restrictions force $\sigma_1 + 1 = \sigma_5$ and $\sigma_1 - 1 = \sigma_3$ or σ_4. Because $\sigma_2 \pm 1 \in \{\sigma_3, \sigma_4, \sigma_5\}$, the only possibility is that $\sigma_1 = k-2$, $\sigma_5 = k-1$, $\sigma_2 = 0$, and $\{\sigma_4, \sigma_3\} = \{1, k-3\}$. As $\sigma_i - \sigma_j \neq 1$ for $6 \leq i < j \leq k$, we conclude that for $i = 6, 7, \ldots, k$, $\sigma_i = i - 4$. As $\sigma_6 - 1 \neq \sigma_3$, we conclude that $\sigma_3 = k-3$ and $\sigma_4 = 1$.

So in the summation below,

$$[R(G)]_{\varphi'}((k-1)!)^{2k} = \sum_{\sigma} \left((-1)^{\sum_{v \in V(G)}(k-1+\sigma(v))} \prod_{v \in V(G)} \binom{k-1}{\sigma(v)} \right) R(G, \sigma),$$

there are only two nonzero entries. Moreover, for these two nonzero entries,

$$\left((-1)^{\sum_{v\in V(G)}(k-1+\sigma(v))}\prod_{v\in V(G)}\binom{k-1}{\sigma(v)}\right)$$

is a nonzero constant. Some straightforward but tedious calculation of this summation of two terms shows that $[R(G)]_{\varphi'} \neq 0$. Hence $[P(G)]_{\varphi} = [Q(G)]_{\varphi} \neq 0$, and G is on-line k-choosable. $\square$

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